NILPOTENCY OF ORTHOGONAL LIE ALGEBRAS $\mathfrak{O}(2l, F)$ OVER FIELDS OF CHARACTERISTIC 2

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Abstract: Classification of nilpotent Lie algebras is an open problem. In this paper, we discuss about nilpotency of Classical Lie Algebras D_l ($l \geq 2$) i.e. Orthogonal Lie Algebras $\mathfrak{O}(2l, F)$ of even dimensions over fields of characteristic two. We show that for $l \geq 2$, $\mathfrak{O}(2l, F)$ is nilpotent when characteristic of F is two. We also show that $\mathfrak{O}(2l, F)$,$l \geq 2$, is not nilpotent when characteristic of F is zero.

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Introduction: Sophus Lie studied certain transformation groups which is the root of Lie theory. His work led to the discovery of Lie groups and Lie algebras. Both Lie groups and Lie algebras have become essential to many parts of mathematics and theoretical physics, in particular, the theory of Relativity. Lie theory has proved to be the key in solving many problems related to Geometry and Differential equations, which links mathematics to real world. Lie algebras arise "in nature" as vector spaces of linear transformations endowed with a new operation which is in general neither commutative nor associative.

Definition (**Lie algebra**) A Lie algebra is a vector space L over a field F, with an operation
$[\ ,\ ] : L \times L \to L$, denoted $(x, y) \mapsto [x, y]$,(called the bracket or commutator of x and y), satisfying the following properties :
(L1) The bracket operation is bilinear.
(L2) $[x, x] = 0$, for all $x \in L$.
(L3) $[x, [y, z]] + [y, [z, x]] + [z,[x, y]] = 0$, for all $x, y, z \in L$.
(L3) is called the Jacobi identity.
$0 = [x + y , x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$
Hence, condition (L1) and (L2) implies (L2') $[x, y] = -[y, x]$ (anticommutativity) for all $x, y \in L$.
If char $F \neq 2$, then putting $x = y$ in (L2'), shows that (L2') implies (L2).

Definition (**Lie Subalgebra**) A subspace K of a Lie algebra L is called a subalgebra if $[x, y] \in K$, whenever $x, y \in K$.

Unless specifically stated, we shall be concerned with Lie algebras L whose underlying vector space is finite dimensional.

Some Examples:

(1) Any vector space V, with $[x, y] = 0$, for all $x, y \in V$ is a Lie algebra called Abelian Lie algebra. In particular, the field F may be regarded as a 1-dimensional abelian Lie algebra.
(2) Let V be a finite dimensional vector space over F with dim $(V) = n$. Let End V be the set of all linear transformations from $V \to V$. This is again a vector space over F of dimension n^2. Define a operation on End V, by $[x, y] = xy - yx$. With this operation End V becomes a Lie algebra over F. End V (also written $gl(V)$) is called General linear algebra. Any subalgebra of a Lie algebra $gl(V)$ is called a linear Lie algebra. The general linear algebra $gl(V)$ can be identified with the set of all $n \times n$ matrices over F, denoted $gl(n, F)$, with the Lie bracket defined by $[x, y] = xy - yx$, where xy is the usual product of the matrices x and y. As a vector space, $gl(n, F)$ has a basis consisting of the matrix units e_{ij} for $1 \leq i, j \leq n$. Here, e_{ij} is the $n \times n$ matrix which has 1 in the (i, $j$ssssth position and 0 elsewhere. As $e_{ij}e_{kl} = \delta_{jk}e_{il}$ it follows that: $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$ where δ is the Kronecker delta, defined by,$\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ if $i \neq j$.

Classification of Lie algebras: There are three different types of Lie Algebras: the semi-simple, the solvable and those which are neither semi-simple nor solvable. Determining the classification of each of these types is equivalent to determine the classification of Lie algebras. The problem of classifying semi-simple Lie algebras is equivalent to that of classifying all non-isomorphic simple Lie algebras. At present, the classification of semi-simple Lie algebras is completely solved. Simple Lie algebras are classified in five different classes, also called Classical Lie algebras: the algebras belonging to special linear group(A_l), the odd orthogonal algebras(B_l), the symplectic algebras(C_l), the even orthogonal algebras(D_l) and the five exceptional Lie algebras E_6, E_7, E_8, F_4 and G_2 which do not belong to any of the previous classes. The splittable algebra reduced the classification of all solvable Lie algebras to the classification of nilpotent Lie algebras. The classification of nilpotent Lie algebras is an open problem. In this paper, we show that for $l \geq 2$, $\mathfrak{O}(2l, F)$ is nilpotent when characteristic of F is two but $\mathfrak{O}(2l, F)$, $l \geq 2$, is not nilpotent when characteristic of F is zero.

Classical Lie algebras D_l: Let dim $V = 2l$ be odd and take f to be the nondegenerate symmetric bilinear form on V whose matrix is $s=\begin{pmatrix}0 & I_l\\ I_l & 0\end{pmatrix}$. The orthogonal algebra $\mathfrak{O}(V)$ or $\mathfrak{O}(2l, F)$ consists of all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$. If we partition x in the same form as s, say $x = \begin{pmatrix}m & n\\ p & q\end{pmatrix}(m, n, p, q \in gl(l,F))$ then the condition $sx = -x^t s$ translates into the following set of conditions: $q = -m^t$, $n^t = -n$, $p^t = -p$. This shows that $Tr(x) = 0$.A basis of D_l can be enumerated as follows:

$e_{i,j} - e_{l+j,l+i}$, $1 \le i, j \le l$; $e_{i,l+j} - e_{j,l+i}$; $e_{l+i,j} - e_{l+j,i}$, $1 \le i < j \le l$;

Where e_{ij} is the matrix having 1 in the (i, j) position and 0 elsewhere.

Hence dim $D_l = l^2 + \frac{1}{2}l(l-1) + \frac{1}{2}l(l-1) = 2l^2 - l$.

Definition (**Ideals**) A subspace I of a Lie algebra L is called an ideal of L if $[x, y] \in I$,for all $x \in L$, $y \in I$. By anticommutativity ($L2'$), ideals are automatically two sided.

Definition (**Derived algebra**) The Derived algebra of L, denoted $[L, L]$ is an ideal of L consisting of all linear combinations of commutators $[x, y]$.

Definition (**Descending central series**) Define a sequence of ideals of L by

$L^0 = L$, $L^1 = [L, L]$, $L^2 = [L, L^1]$, ... , $L^{i+1} = [L, L^i]$.

The sequence of ideals defined above is called the descending central series or the lower central series.

Definition (**Nilpotent**) L is called nilpotent if $L^n = 0$ for some n.

Now we show that for l ≥ 2, $\mathfrak{O}(2l, F)$ is nilpotent when characteristic of F is 2.:

The standard basis for D_l can be enumerated as follows:

$e_{ij} - e_{l+j,l+i}$, $1 \le i, j \le l$; $e_{i,l+j} - e_{j,l+i}$; $e_{l+i,j} - e_{l+j,i}$, $1 \le i < j \le l$.

For $i \neq j$, we have,

- $[e_{ij} - e_{l+j,l+i}, e_{ji} - e_{l+i,l+j}] + [e_{i,l+j} - e_{j,l+i}, e_{l+j,i} - e_{l+i,j}]$

$= ([e_{ij}, e_{ji}] - [e_{ij}, e_{l+i,l+j}] - [e_{l+j,l+i}, e_{ji}] + [e_{l+j,l+i}, e_{l+i,l+j}])$
$+ ([e_{i,l+j}, e_{l+j,i}] - [e_{i,l+j}, e_{l+i,j}] - [e_{j,l+i}, e_{l+j,i}] + [e_{j,l+i}, e_{l+i,j}])$
$= ((\delta_{jj}e_{ii} - \delta_{ii}e_{jj}) - (\delta_{j,l+i}e_{i,l+j} - \delta_{l+j,i}e_{l+i,j}) - (\delta_{l+i,j}e_{l+j,i} - \delta_{i,l+j}e_{j,l+i}) +$
$(\delta_{l+i,l+i}e_{l+j,l+j} - \delta_{l+j,l+j}e_{l+i,l+i})) + ((\delta_{l+j,l+j}e_{ii} - \delta_{ii}e_{l+j,l+j}) -$
$(\delta_{l+j,l+i}e_{ij} - \delta_{ji}e_{l+i,l+j}) - (\delta_{l+i,l+j}e_{ji} - \delta_{ij}e_{l+j,l+i}) + (\delta_{l+i,l+i}e_{jj} - \delta_{jj}e_{l+i,l+i}))$
$= ((e_{ii} - e_{jj}) + (e_{l+j,l+j} - e_{l+i,l+i})) + ((e_{ii} - e_{l+j,l+j}) + (e_{jj} - e_{l+i,l+i}))$
$= 2e_{ii} - 2e_{l+i,l+i}$
$=2(e_{ii} - e_{l+i,l+i})$
$= 0.$

- $[e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}]$

$= [e_{ii}, e_{ij}] - [e_{ii}, e_{l+j,l+i}] - [e_{l+i,l+i}, e_{ij}] + [e_{l+i,l+i}, e_{l+j,l+i}]$
$= (\delta_{ii}e_{ij} - \delta_{ji}e_{ii}) - (\delta_{i,l+j}e_{i,l+i} - \delta_{l+i,i}e_{l+j,i}) - (\delta_{l+i,i}e_{l+i,j} - \delta_{j,l+i}e_{i,l+i})$
$+ (\delta_{l+i,l+j}e_{l+i,l+i} - \delta_{l+i,l+i}e_{l+j,l+i})$
$= e_{ij} - e_{l+j,l+i}$

- $[e_{ii} - e_{l+i,l+i}, e_{i,l+j} - e_{j,l+i}]$

$= [e_{ii}, e_{i,l+j}] - [e_{ii}, e_{j,l+i}] - [e_{l+i,l+i}, e_{i,l+j}] + [e_{l+i,l+i}, e_{j,l+i}]$
$= (\delta_{ii}e_{i,l+j} - \delta_{l+j,i}e_{ii}) - (\delta_{ij}e_{i,l+i} - \delta_{l+i,i}e_{ji}) - (\delta_{l+i,i}e_{l+i,l+j} - \delta_{l+j,l+i}e_{i,l+i})$
$+ (\delta_{l+i,j}e_{l+i,l+i} - \delta_{l+i,l+i}e_{j,l+i})$
$= e_{i,l+j} - e_{j,l+i}.$

- $[e_{l+i,l+i} - e_{ii}, e_{l+i,j} - e_{l+j,i}]$

$= [e_{l+i,l+i}, e_{l+i,j}] - [e_{l+i,l+i}\ e_{l+j,i}] - [e_{ii}, e_{l+i,j}] + [e_{ii}, e_{l+j,i}]$
$= (\delta_{l+i,l+i}e_{l+i,j} - \delta_{j,l+i}e_{l+i,l+i}) - (\delta_{l+i,l+j}e_{l+i,i} - \delta_{i,l+i}e_{l+j,l+i})$
$- (\delta_{i,l+i}e_{ij} - \delta_{ji}e_{l+i,i}) + (\delta_{i,l+j}e_{ii} - \delta_{ii}e_{l+j,i})$
$= e_{l+i,j} - e_{l+j,i}.$

$Hence, \mathfrak{O}(2l, F)^0 = \mathfrak{O}(2l, F)$

$\mathfrak{O}(2l, F)^1 = [\mathfrak{O}(2l, F), \mathfrak{O}(2l, F)]$
$= F\{e_{ij} - e_{l+j,l+i}, e_{i,l+j} - e_{j,l+i}, e_{l+i,j} - e_{l+j,i}\}.$

$\mathfrak{O}(2l, F)^2 = [\mathfrak{O}(2l, F)^1, \mathfrak{O}(2l, F)^1]$
$= [F\{e_{ij} - e_{l+j,l+i}, e_{i,l+j} - e_{j,l+i}, e_{l+i,j} - e_{l+j,i}\},$
$F\{e_{ij} - e_{l+j,l+i}, e_{i,l+j} - e_{j,l+i}, e_{l+i,j} - e_{l+j,i}\}]$
$= 0.$

Hence, for $l \ge 2$, $\mathfrak{O}(2l, F)$ is nilpotent when characteristic of F is 2.

Now we show that for l ≥ 2, $\mathfrak{O}(2l, F)$ is not nilpotent when characteristic of F is zero.:

For $i \neq j$, we have,

- $\frac{1}{2}[e_{ij} - e_{l+j,l+i}, e_{ji} - e_{l+i,l+j}] + \frac{1}{2}[e_{i,l+j} - e_{j,l+i}, e_{l+j,i} - e_{l+i,j}]$

$= e_{ii} - e_{l+i,l+i} \in [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)]$.

- $[e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}] = e_{ij} - e_{l+j,l+i} \in [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)]$.
- $[e_{ii} - e_{l+i,l+i}, e_{i,l+j} - e_{j,l+i}] = e_{i,l+j} - e_{j,l+i} \in [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)]$.
- $[e_{l+i,l+i} - e_{ii}, e_{l+i,j} - e_{l+j,i}] = e_{l+i,j} - e_{l+j,i} \in [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)]$.

Hence, $\mathfrak{O}(2l,F) = [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)]$.

$\mathfrak{O}(2l,F)^0 = \mathfrak{O}(2l,F)$

$\mathfrak{O}(2l,F)^1 = [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)] = \mathfrak{O}(2l,F)$.

$\mathfrak{O}(2l,F)^2 = [\mathfrak{O}(2l,F)^1, \mathfrak{O}(2l,F)^1] = [\mathfrak{O}(2l,F), \mathfrak{O}(2l,F)] = \mathfrak{O}(2l,F)$.

In general, $\mathfrak{O}(2l,F)^m = \mathfrak{O}(2l,F)$, for all $m \in \mathbb{Z}^+$.

Hence, for $l \geq 2$, $\mathfrak{O}(2l,F)$ is not nilpotent when characteristic of F is zero.

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