FIXED POINT RESULTS FOR ALTERING DISTANCE FUNCTIONS

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Abstract: In this paper we proves a generalised results of J.R. Morales , E.M. Rojas , B.K. Dasand, S. Gupta .Also the results given by B. Samet and H. Yazid using altering distance functions and property P for the contraction mappings.

Keywords: Fixed point, Altering distance functions, Complete metric space.

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Introduction and Preliminaries: The fixed point theorems in metric spaces are playing a major role to solve many problems in a mathematical analysis. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

Altering distance function for self-mapping on a metric space established by M.S. Khan in 1984 and it can be expanded by M. Swalesh, S. Sessa that they introduced a control function which they called as altering distance function in the research of fixed point theory. The author Mier- Keeler type (ε, δ)- contractive condition to study of fixed point by using a control function with extended contractive conditions.

Definition 1 A function $\psi: \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ is called an altering distance function if the following properties are satisfied.

(φ_1) $\psi(t) = 0 \Leftrightarrow t = 0$.

(φ_2) ψ is monotonically non decreasing.

(φ_3) ψ is continuous.

By ψ wedenotes the set of all altering distance function.

Using those control functions the author extend the Banach contraction principle by taking $\psi = \text{Id}$, (the identity mapping), in the inequality contraction (1.1) of the following theorem.

Theorem1.1 *Let (M, d)be a complete metric space, let $\psi \in \Psi$ and let $Q : M \to M$ be a mapping which satisfies the following inequality*

$\psi[d(Q_x, Q_y)] \leq a\psi[d(x, y)]$ [1.1]

for all $x, y \in M$ and for some $0 < a < 1$. Then , shas a unique fixed point $v_0 \in M$ and moreover for each $x \in M$, $\lim_{n\to\infty} Q^n x = v_0$.

Fixed point theorems involving the notion of altering distance functions has been widely studied, On the other hand, in 1975, B.K. Das and S. Gupta [3] proves the following result.

Theorem1.2*Let (M, d)be a metric space and let $Q: M \to M$ be a given mapping such that,*

(i) $d(Qx, Qy) \leq \alpha\, d(x, y) + \beta\, m(x, y)$ [1.2]

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < 1$ where

$m(x, y) = \left[\frac{d^2(x,Qx) + d(x,Qy)\, d(y,Qx) + d^2(y,Qy)}{1 + d(x,Qx) d(y,Qy)}\right]$[1.3]

for all $x, y \in M$.

(ii) *for some $x_0 \in M$, the sequence of iterates $(Q^n x_0)$has a subsequence $(Q^{nk} x_0)$*

Withlim$_{k\to\infty} Q^{nk} x_0 = v_0$ *.Then v_0 is the unique fixed point of Q.*

Definition1.2. Let (M, d) be a metric space for a self-mapping Q with a nonempty fixed point set E(Q). Then Q is said to satisfy the property P If E(Q) = E(Q^n) for each n∈ N.

Lemma 1.3. Let (M, d) be a metric space. Let $\{y_n\}$ be a sequence in M such that

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ [1.4]

If $\{y_n\}$ is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequence of integers positive (m(k)) and (n(k)) with

(m(k))> (n(k)) $> k$, such that,

$d\left(y_{(\mathrm{m(k)})}, y_{(\mathrm{n(k)})}\right) \geq \varepsilon_0$, $d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})}\right) < \varepsilon_0$, and

1. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})+1}\right) = \varepsilon_0$
2. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})}, y_{(\mathrm{n(k)})}\right) = \varepsilon_0$
3. $\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})-1}, y_{(\mathrm{n(k)})}\right) = \varepsilon_0$

Remark 1.4. From Lemma 1.3 is easy to get

$$\lim_{k\to\infty} d\left(y_{(\mathrm{m(k)})+1}, y_{(\mathrm{n(k)})+1}\right) = \varepsilon_0$$

In this paper we will study the property introduced by G.S. Jeong and B.E. Rhoades in [5] which they called the property P in metric spaces

Main Result

Theorem 2.1 Let a complete metric space (M, d) , we have $\psi \in \Psi. Let\ Q : M \to M$ be a mapping which satisfies the condition:

$$\psi[d(Qx,Qy)] \le \alpha\, \psi[d(x,y)] + \beta\, \psi\left[\frac{d^2(x,Qx)+d(x,Qy)\, d(y,Qx)+d^2(y,Qy)}{1+d(x,Qx)d(y,Qy)}\right] \qquad [2.1]$$

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + 2\beta < 1\ and\ m(x,y)$ is given by [1.2]. Then Q has a unique fixed point $v_o \in M$, and for each $x \in M$ $\lim_{n\to\infty} Q^n\, x = v_o.$

Proof: Let $x \in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as:

$$x_{n+1} = Qx_n = Q^{n+1}x$$

For all n ≥1, Now

$$\psi[d(x_n, x_{n+1})] = \psi[d(Qx_{n-1}, Qx_n)] \qquad [2.2]$$

$$\le \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, Qx_{n-1}) + d(x_{n-1}, Qx_n)\, d(x_n, Qx_{n-1}) + d^2(x_n, Qx_n)}{1 + d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right]$$

$$\psi[d(x_n, x_{n+1})] \le \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, Qx_{n-1})}{1 + d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right]$$

$$+ \beta\, \psi\left[\frac{+d(x_{n-1},Qx_n)\, d(x_n,Qx_{n-1})}{1+d(x_{n-1},Qx_{n-1})d(x_n,Qx_n)}\right] + +\beta\, \psi\left[\frac{d^2(x_n,Qx_n)}{1+d(x_{n-1},Qx_{n-1})d(x_n,Qx_n)}\right]$$

$$\le \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)d(x_n, x_{n+1})}\right]$$

$$+ \beta\, \psi\left[\frac{+d(x_{n-1},x_{n+1})\, d(x_n,x_n)}{1+d(x_{n-1},x_n)d(x_n,x_{n+1})}\right] + +\beta\, \psi\left[\frac{d^2(x_n,x_{n+1})}{1+d(x_{n-1},x_n)d(x_n,x_{n+1})}\right]$$

$$\le \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\psi[d(x_n, x_{n+1})] \le (\alpha + \beta)\psi[d(x_{n-1}, x_n)] + \beta\, \psi d(x_n, x_{n+1})$$

$$\psi[d(x_n, x_{n+1})] \le (\alpha + \beta)\psi[d(x_{n-1}, x_n)]$$

$$\psi[d(x_n, x_{n+1})] \le \frac{(\alpha + \beta)}{(1-\beta)}\psi[d(x_{n-1}, x_n)]$$

$$\le \left[\frac{(\alpha + \beta)}{(1-\beta)}\right]^2 \psi[d(x_{n-2}, x_{n-1})] \le \_\_\_\_$$

$$\psi[d(x_n, x_{n+1})] \le \left[\frac{(\alpha+\beta)}{(1-\beta)}\right]^n \psi[d(x_0, x_1)] \qquad [2.3]$$

since$\frac{\alpha}{1-\beta} \in (0,1)$ from (3), we obtain

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$$

From the result given that $\psi \in \Psi$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \qquad [2.4]$$

Now, we will show that (x_n) is Cauchy sequence in M. Suppose that (x_n) is not a Cauchy sequence, which means that there is a constant $\in > 0$ such that for each positive integer k, there exist a positive integer m(k)and n(k) with m(k)>n(k)>k such that

$d(x_{m(k)}, x_{n(k)}) \ge \in_0$, $d(x_{m(k)-1}, x_{n(k)}) < \in_0$

From lemma 1.3 and remark 1.4 we have,

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \in_0 \qquad [2.5]$$

And $\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \in_0$ [2.6]

For $x = x_{m(k)} and\ y = y_{n(k)}$ from [1] we have

$$d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \psi\left[d(Qx_{m(k)}, x_{n(k)})\right]$$

$$\le \alpha\, \psi\left[d\left(x_{m(k)}, x_{n(k)}\right)\right] + \beta\, \psi\left[\frac{d^2\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{n(k)+1}\right) d\left(x_{n(k)}, x_{n(k)}\right) + d^2\left(x_{n(k)}, x_{m(k)+1}\right)}{1 + d(x_{m(k)}, x_{n(k)})d(x_{n(k)}, x_{n(k)+1})}\right]$$

Using [4], [5] and [6] we have

$$\psi(\in) = \lim_{k\to\infty} \beta\, \psi\left[d\left(x_{n(k)}, x_{n(k)+1}\right)\right] \le \beta \lim_{k\to\infty} \psi\left[d\left(x_{n(k)-1}, x_{n(k)}\right)\right]$$

$$\le \beta \lim_{k\to\infty} \psi\left[d\left(x_{m(k)}, x_{n(k)}\right)\right]$$

$$\le \alpha\, \psi(\in)$$

Since $\alpha \in (0,1), we\ get\ a\ contradiction$. Thus $\{x_n\}$ is a Cauchy sequence in the complete metric space M, Thus there exist $v_0 \in M$such that

$$\lim_{n\to\infty} x_n = v_0$$

Setting $x = x_n$ *and* $y = v_0$ *in* [1], *we have*

$$\psi[d(x_{n+1}, Qv_0)] = \psi[d(Qx_n, Tv_0)]$$
$$\leq \alpha\ \psi[d(x_n, v_0)] + \beta\ \psi\left[\frac{d^2(x_n, Qx_n) + d(x_n, Qv_0)\ d(v_0, Qx_n) + d^2(v_0, Qv_0)}{1 + d(x_n, Qx_n) + d(v_0, Qv_0)}\right]$$

Therefore $\lim_{n\to\infty} \psi[d(x_{n+1}, Qv_0)] \leq \beta\ \psi\ d(v_0, Qv_0)$

$$i.e. \qquad \psi\ d(v_0, Qv_0) \leq \beta\ \psi\ d(v_0, Qv_0)$$

since $\beta \in (0,1)$, *then* $\psi\ d(v_0, Qv_0) = 0$, *which implies that* $d(v_0, Qv_0) = 0$
thus $v_0 = Qv_0$.

Now we are going to establish the uniqueness of the fixed point, Lety_0, v_0 be two fixed point of Q such that $y_0 \neq v_0$, *putting* $x = y_0$ *and* $y = v_0$ *in* [1], we get

$$\psi\ d(Qv_0, Qy_0) \leq \alpha\ \psi[d(v_0, y_0)]$$
$$+\beta\ \psi\left[\frac{d^2(v_0, Qv_0) + d(v_0, Qy_0)\ d(y_0, Qv_0) + d^2(y_0, Qy_0)}{1 + d(v_0, Qv_0) + d(y_0, Qy_0)}\right]$$

i. e. $\psi\ d(Qv_0, Qy_0) \leq \alpha\ \psi[d(v_0, y_0)]$
whichimplies that $\psi[d(v_0, y_0)] = 0$, *so* $d(v_0, y_0) = 0$
Thus$v_0 = y_0$.

Corollary 2.2 Let (M, d) be a complete metric space and let Q : M → M be a mapping. We assume that for each $x, y \in M$,

$$\int_0^{d(Qx,Qy)} \psi(u)du \leq \alpha \int_0^{d(x,y)} \psi(u)du + \beta \int_0^{\psi\left[\frac{d^2(v_0,Qv_0)+d(v_0,Qy_0)\ d(y_0,Qv_0)+d^2(y_0,Qy_0)}{1+d(v_0,Qv_0)+d(y_0,Qy_0)}\right]} \psi(u)du \quad [2.7]$$

Where o< $\alpha + \beta < 1$ *and* $\psi : R_+ \to R_+$ is a Lebesgue integrable mapping which is summable on each compact subset of [0, +∞) , *non negative and such that*SS

$$\int_0^{\epsilon} \psi(u)du > 0, \quad for\ all\ \epsilon > 0.$$

Then Q has a unique fixed point $v_0 \in M$ *such that for each* $x \in M$, $\lim_{n\to\infty} Q^n x = v_0$.

Proof: Let $\psi : R_+ \to R_+$be a mapping as we define $\psi_0(u) = \int_0^u \psi(u)du, u \in R_+$. It is clear that $\psi_0(0) = 0$. $\psi_0$is monotonically non decreasing and by hypothesis $\Psi_0$is absolutely continuous. Hence $\psi_0$is continuous. Therefore, $\psi_0 \in \Psi$, so by (2.1)becomes

$$\psi_0(d(Qx, Qy)) \leq \alpha\psi_0(d(x, y)) + \beta\ \psi_0\left[\frac{d^2(v_0, Qv_0) + d(v_0, Qy_0)\ d(y_0, Qv_0) + d^2(y_0, Qy_0)}{1 + d(v_0, Qv_0) + d(y_0, Qy_0)}\right]$$

Hence from theorem 2.1 there exists a unique fixed point $v_0 \in M$ *such that for each* $x \in M$, $\lim_{n\to\infty} Q^n x = v_0$.

Remarks 2.3.

1. If we take $\beta = 0$, then (2.1) reduces to (1.2), thus the Theorem 1.1 is a corollary of theorem 2.1.
2. If we take $\psi = I\rho$ in (2.1), then we obtain (1.2). Therefore the Theorem 2.1is a generalisation of Theorem 1.2.

3 The property P.

In this section we are going to prove that the mappings satisfying the contractive conditions [1.1], [1.2], [2.1] and [2.7] fulfil the property P.

Theorem 3.1 Let (M, d) be a completemetric space, we have $\psi \in \Psi$. *Let* $Q : M \to M$ be a mapping which satisfies the condition:

$$\psi[d(Qx, Qy)] \leq \alpha\ \psi[d(x, y)]$$

for all $x, y \in M$, *and for some* $0 < \alpha < 1$. *Then* $E_Q \neq \phi$ and Q has a property P.

Proof: From Theorem [1.1], Q has a fixed point therefore $E_{Q^n} \neq \phi$ for every n ∈ N,
Fix $n > 1$and we assume that$v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that $v \neq Qv$, *from* [1.1]

$\psi[d(v, Qv)] = \psi[d(Q^n v, Q^{n+1} v)] \leq a\psi[d(Q^{n-1} v, Q^n v)] \leq \ldots\ldots \leq a^n \psi[d(v, Qv)]$.

Since a ∈ (0,1), $\lim_{n\to\infty} \psi[d(v, Qv)] = 0$. From the fact that, $\psi \in \Psi$ we get $v = Qv$ which is a contradiction. Therefore $v \in E_Q$ i.e. Q has a property P.

Theorem 3.2 Let (M, d) be a complete metric space, and *Let* $Q : M \to M$ be a mapping which satisfies the contractive condition:

$$\psi[d(Qx, Qy)] \leq \alpha\ [d(x, y)] + \beta\ m(x, y)$$

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < 1$ *where*

$$m(x,y) = \left[\frac{d^2(x,Qx) + d(x,Qy)\, d(y,Qx) + d^2(y,Qy)}{1 + d(x,Qx)d(y,Qy)}\right]$$

$Then\ E_Q \neq \phi$and Q has a property.

Proof: From Theorem [1.2], $E_Q \neq \phi$,therefore $E_{Q^n} \neq \phi$ for every n $\in N$,

$Fix\ n > 1$and we assume that $v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that

$v \neq Qv$

$$d(v,Qv) = d(Q^n v, Q^{n+1}v)$$
$$\leq ad(Q^{n-1}v, Q^n v) + b\left[\frac{d^2(Q^{n-1}v, Q^n v) + d(Q^{n-1}v, Q^{n+1}v)d(Q^n v, Q^n v) + d^2(Q^n v, Q^{n+1}v)}{1 + d(Q^{n-1}v, Q^n v) + d(Q^n v, Q^{n+1}v)}\right]$$
$$= ad(Q^{n-1}v, Q^n v) + bd(Q^n v, Q^{n+1}v)$$

Therefore $d(v,Qv) = d(Q^n v, Q^{n+1}v) \leq \frac{a}{1-b}\, d(Q^{n-1}v, Q^n v) \leq \ldots. \leq \left(\frac{a}{1-b}\right)^n d(v,Qv)$

Which is a contradiction. Consequently $v \in E_Q$ and Q has the property P.

Theorem 3.3 Let (M, d) be a complete metric space, let ψ ∈ Ψ and $Let\ Q : M \to M$ be a mapping which satisfies the contractive condition:

$$\psi[d(Qx,Qy)] \leq \alpha\, \psi[d(x,y)] + \beta\psi\left[\frac{d^2(x,Qx) + d(x,Qy)\, d(y,Qx) + d^2(y,Qy)}{1 + d(x,Qx)d(y,Qy)}\right]$$

$Then\ E_Q \neq \phi$and Q has a property P.

Proof: From Theorem [1.1], Q has a fixed point therefore $E_{Q^n} \neq \phi$ for every n $\in N$,

$Fix\ n > 1$and we assume that $v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that

$v \neq Qv, from$ [2.1]

$$\psi[d(v,Qv)] = \psi\,[d(Q^n v, Q^{n+1}v)]$$
$$\leq a\,\psi[d(Q^{n-1}v, Q^n v)]$$
$$+ b\,\psi\left[\frac{d^2(Q^{n-1}v, Q^n v) + d(Q^{n-1}v, Q^{n+1}v)d(Q^n v, Q^n v) + d^2(Q^n v, Q^{n+1}v)}{1 + d(Q^{n-1}v, Q^n v) + d(Q^n v, Q^{n+1}v)}\right]$$
$$= a\,\psi d(Q^{n-1}v, Q^n v) + b\,\psi d(Q^n v, Q^{n+1}v)$$

Hence $\psi\, d(v,Qv) = \psi\, d(Q^n v, Q^{n+1}v) \leq \frac{a}{1-b}\psi\, d(Q^{n-1}v, Q^n v) \leq \ldots. \leq \left(\frac{a}{1-b}\right)^n \psi d(v,Qv)$

$$\psi d(v,Qv) \leq \left(\frac{a}{1-b}\right)^n \psi d(v,Qv)$$

Which is a contradiction, thereforeψ $d(v,Qv) = 0, since\ \psi \in \Psi$

We conclude that $d(v,Qv) = 0, thus\ v \in E_Q$ and Q has the property P.

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