VIBRATIONS ANALYSIS OF THERMOELASTIC SPHERICAL CURVED PLATES IN CIRCUMFERENTIAL DIRECTIONS

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Abstract: This paper deals with the study of free vibration analysis of toroidal and spheroidal thermoelastic vibrations of spherical curved plates. The spherical curved plate is assumed to be stress free in equilibrium state. The ordinary differential equations have been obtained by reducing partial differential equations using time harmonics vibrations. The uncoupled equation which is independent of temperature is taken for toroidal vibrations and coupled system of equations which depend upon temperature is taken for spheroidal vibrations. Matrix Fröbenious method of series solution has been applied in the coupled system of ordinary differential equations to obtained displacements, stresses and temperature. The convergence analysis has been applied which shows that the series can be differentiated term by term and being analytic in nature. The secular equations have been obtained by applying assumed boundary conditions. To represent numerical results the fixed point iteration numerical technique has been implemented to obtain eigen field quantities with the help of MATLAB software tools. Numerical results have been presented graphically for lowest frequency, dissipation factor.

Keywords: Matrix Fröbenius Method; Time harmonics; Toroidal; Spheroidal; Lowest frequency; Dissipation factor.

1. Introduction: The uncoupled classical theory of thermo elasticity predicts two phenomena which are not compatible with physical observations. First, the heat conduction equation, this theory does not contain any elastic terms. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves. Dhaliwal and Singh [1] studied the treatment of such problems of vibration of isotropic and anisotropic spherical structures in detail. Lamb [2] viewed the existence of two types of vibrations namely; (i) zero volume change and zero radial displacement are the toroidal vibrations and (ii) the spheroidal vibrations with zero radial components of the curl of the displacement vector. Sâto and Usami [3] studied the natural frequency parameters for the modes of vibration for the solid sphere. Chen et al. [4] investigated the behaviour of eigen frequencies of anisotropic sphere. Cohan et al. [5] studied the free vibrations of spherically isotropic hollow sphere in the context of coupled elasticity. Lord and Schulman [6] have modified the Fourier law of heat conduction in the context of generalized thermo elasticity. Green and Lindsay [7] have formulated temperature rate dependent thermo elasticity by modifying law of entropy. The plane waves in visco thermo elasticity have been investigated by Othman et al. [8] in the context of generalized theory of thermo elasticity. Jiangong and Qiujuan [9] studied waves propagating in non – homogenous magneto-electro-elastic spherical curved plates composed of piezoelectric and magneto elastic materials. Towfighi et al. [10] studied the importance of pipe and cylindrical pressure vessel inspections in circumferential directions. Sharma and Sharma [11] investigated the vibrations of a transradially isotropic coupled thermoplastic solid sphere. Towfighi and Kundu [12] solved the guided waves in spherical curved plates for both isotropic and anisotropic materials.

In this paper the equations of motions and heat conduction have been solved with the help of Matrix Fröbenius Method of series solution to obtain displacements, stresses and temperature change. Computer simulated results have been shown graphically.

2. Formulation of Problem: Free vibration analysis of homogenous isotropic thermally insulated and isothermal thermoelastic spherical curved plate having outer radius, inner radius and thickness be a , b , h . Let the plate being assumed to be initially at undisturbed state at uniform temperature T_0 . According to Towfighi and Kundu [12] the wave carrier has been interchangeably called a curved plate; spherical plate, pipe segment or simply pipe, all meaning the same thing. According to Towfighi and Kundu [12] the wave propagation in from point A to B is a spherical plate segment, the two points A and B of a sphere by adjusting the positions of the north and south poles can always be aligned along the equator. To study the wave propagation between two points in a spherical curved plate segment, it is sufficient to solve the governing equations for $\theta = \pi/2$ only and wave propagation is independent of θ . The basic equations of motion and heat conduction for displacement $\vec{u}(r, \phi, t) = (u_r, u_\theta, u_\phi)$ and temperature change $T(r, \phi, t)$ in the absence of body forces and heat sources in tensor form are given by (Towfighi and Kundu [12] and Dhaliwal and Singh [1])

$$\sigma_{ij,j} = \rho \ddot{u} \tag{1}$$

$$K T_{,ij} - \rho C_e \left(\dot{T} + t_0 \ddot{T} \right) = T_0 \beta \left[\dot{e} + t_0 \delta_{1k} \ddot{e} \right] \tag{2}$$

where $e = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}$, $\beta = (3\lambda + 2\mu)\alpha_T$,

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\left(2\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} \right) - \beta\left(T + t_1 \delta_{2k} \dot{T} \right)$$

$$\sigma_{\theta\theta} = (\lambda + 2\mu)\frac{u_r}{r} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} \right) - \beta\left(T + t_1 \delta_{2k} \dot{T} \right)$$

$$\sigma_{\phi\phi} = (\lambda + 2\mu)\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} \right) + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) - \beta\left(T + t_1 \delta_{2k} \dot{T} \right)$$

$$\left\{ \sigma_{r\theta}, \sigma_{r\phi}, \sigma_{\theta\phi} \right\} = \left\{ \begin{matrix} \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \\ \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r} \right), \ \mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \phi} \right) \end{matrix} \right\} \tag{3}$$

Here σ_{ij}, $(i, j = r, \phi)$ are the stress components; β is the thermoelastic coupling constant; α_T is the coefficient of linear thermal expansion; ρ is mass density; λ, μ are Lame's parameters; K is the thermal conductivity; C_e is the specific heat at constant strain; t_0 and t_1 are the thermal relaxation times. The quantity e is the dilatation component. The quantity δ_{ik} , $i = 1, 2$ is Kronecker's delta in which $k = 1$ is taken Lord – Shulman (LS) and $k = 2$ refers to Green – Lindsay (GL) theory of thermoelasticity. The comma notion is used for spatial derivatives.

Boundary Conditions of the curved plate which is subjected to stress free, thermally insulated and isothermal conditions (at $r = a$ and $r = b$) are considered. Mathematically this gives us

$$\sigma_{rr} = 0, \ \sigma_{r\theta} = 0, \ \sigma_{r\phi} = 0 \ , \ T_{,r} = 0 \tag{4.1}$$

$$\sigma_{rr} = 0, \ \sigma_{r\theta} = 0, \ \sigma_{r\phi} = 0 \ , \ T = 0 \tag{4.2}$$

3. Solution of the Problem: To remove the complexity of the equations we define following dimensionless quantities

$$\zeta = \frac{r}{R}, \ \tau = \frac{c_1}{R}t, \ \left(U_\zeta, U_\theta, U_\phi \right) = \left(\frac{u_r}{R}, \frac{u_\theta}{R}, \frac{u_\phi}{R} \right), \quad \Theta = \frac{T}{T_0},$$

$$\tau_{ij} = \frac{\sigma_{ij}}{\rho c_1^2}, \ \bar{\beta} = \frac{\beta T_0}{\lambda + 2\mu} \ , \ \varepsilon_T = \frac{T_0 \beta^2}{\rho C_e (\lambda + 2\mu)} \ ,$$

$$\Omega^* = \frac{\omega^* R}{c_1} \ , \ \delta^2 = \frac{c_2^2}{c_1^2} \ , \ R = \frac{a+b}{2}, \tag{5}$$

where $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$ are longitudinal, shear wave velocities and $\omega^* = C_e(\lambda + 2\mu)/K$ is characteristic frequency of the medium, ε_T is thermo-mechanical coupling constant. We introduce the potential ψ , G and w by taking $\theta = \frac{\pi}{2}$ defined by (Chen et al. [4])

$$U_\theta = -\frac{\partial \Psi}{\partial \phi}, \ U_\phi = -\frac{\partial G}{\partial \phi} \ , U_\zeta = w \ ; \tag{6}$$

The wave solution of the form as under Towfighi and Kundu [12]

$$\left\{ \psi, \ w, \ G, \ \Theta \right\} (\zeta, \phi, t) = \zeta^{-\frac{1}{2}} \times \sum_{n=1}^{\infty} \Psi_m(\zeta), \ w_m(\zeta), \ G_m(\zeta), \ \Theta_m(\zeta) \ e^{-i(\Omega\tau + m\phi)} \tag{7}$$

Using non dimensional quantities equation (5) in equations (1) to (3) via (6) and (7) we get

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - \frac{b_3^2}{\xi^2} + \Omega^2 \right) w_m + \left(m^2 \left[(1-\delta^2)\frac{1}{\xi}\frac{\partial}{\partial \xi} - \frac{b_1^2}{\xi^2} \right] \right) G_m + \left(i\Omega\bar{\beta}\tau_1 \left(\frac{\partial}{\partial \xi} - \frac{1}{2\xi} \right) \right) \Theta_m = 0$$

$$-\left[(1-\delta^2)\frac{1}{\xi}\frac{\partial}{\partial \xi} + \frac{b_1^2}{\xi^2} \right] w_m + \left(\delta^2 \left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} \right) - \frac{b_2^2}{\xi^2} + \Omega^2 \right) G_m - \left(i\,\Omega\bar{\beta}\tau_1 \frac{\Theta_m}{\xi} \right) \Theta_m = 0$$

$$c_4^* \left(\frac{\partial}{\partial \xi} + \frac{3}{2\xi} \right) w_m + \frac{c_4^* m^2}{\xi} G_m + \left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - \frac{b_4^2}{\xi^2} + \tau_0' \right) \Theta_m = 0 \tag{8}$$

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + \frac{\Omega^2}{\delta^2} - \frac{\eta^2}{\xi^2} \right) \Psi_m = 0 \tag{9}$$

where $\xi = \zeta\Omega$, $c_4^* = \dfrac{\varepsilon_T\,\Omega^2\tau_0}{\bar{\beta}}$, $b_1^2 = \dfrac{3+\delta^2}{2}$,

$$b_2^2 = \frac{9}{4}\delta^2 + m^2 \ , \ b_3^2 = \frac{9}{4} + m^2\,\delta^2 \ , b_4^2 = \frac{1}{4} + m^2,$$

$$\eta^2 = \frac{9}{4} + m^2 \ , \ i\Omega^{-1} + t_0\delta_{1k} = \tau_0' \ ,$$

$$i\Omega^{-1} + t_1\delta_{2k} = \tau_1 \ , \ i\Omega^{-1} + t_0 = \tau_0 \ ,$$

The uncoupling of equation (9) for ψ_m leads to the toroidal modes of vibrations which remains independent of temperature and can be discussed in the same manner as was done by Cohen et al. [5].

$$\psi_m(\xi) = \sum_{m=1}^{\infty}\left[B_{m1}\ J_\eta\left(\frac{\xi}{\delta}\right) + B_{m2}\ Y_\eta\left(\frac{\xi}{\delta}\right)\right] \qquad (10)$$

where B_{m1} and B_{m2} are arbitrary constants determined from arbitrary conditions.

4. Matrix Fröbenius method: We apply Matrix Fröbenius method in equation (8) to solve coupled system of equations. Clearly the point $\xi = 0$ is a regular singular point of equations (8) and therefore all the coefficients are continuous, single valued and finite in the interval $\xi_2 \le \xi \le \xi_1$, where $\xi_1 = a\Omega$ and $\xi_2 = b\Omega$. We take power series of the type

$$\mathrm{X_n} = \sum_{k=0}^{\infty} \mathbf{Y_k}\,\xi^{s+k} \qquad (11)$$

where

$$X_n = [w_m \quad G_m \quad \Theta_m]' \ , \ \mathbf{Y_k} = [\mathbf{A_k} \quad \mathbf{B_k} \quad \mathbf{C_k}]'$$

s is the eigen value and $\mathbf{A_k}, \mathbf{B_k}, \mathbf{C_k}$ are unknowns to be determined. Domain of the solution is $b \le \zeta \le a$, $a > 0$. The solution (11) is valid in some deleted interval $0 < \zeta < \gamma$, $\gamma > a$ (about the origin), here γ is the radius of convergence. Substituting the solution (11) in equations (8), we get the following matrix equations

$$\sum_{k=0}^{\infty}\left[\mathbf{G_1}(s+k)\xi^{-2} + \mathbf{G_2}(s+k)\xi^{-1} + \mathbf{G}\right]\xi^{s+k}\mathbf{Y_k} = 0 \qquad (12)$$

where

$$\mathbf{G_1}(s+k) = \begin{bmatrix} (s+k)^2 - b_3^2 & m^2\left((1-\delta^2)(s+k) - b_1^2\right) & 0 \\ -((1-\delta^2)(s+k) + b_1^2) & \delta^2(s+k)^2 - b_2^2 & 0 \\ (s+k)^2 - b_4^2 & 0 & 0 \end{bmatrix}, \qquad (13)$$

$$\mathbf{G_2}(s+k) = \begin{bmatrix} 0 & 0 & i\Omega\bar{\beta}\hat{\tau}_1(s+k-1/2) \\ 0 & 0 & -i\Omega\bar{\beta}\hat{\tau}_1 \\ c_4^*(s+k+3/2) & c_4^* m^2 & 0 \end{bmatrix},$$

$$\mathbf{G} = diag\ (\Omega^2 ,\ \Omega^2 ,\ \tau_0) \qquad (14)$$

Equating to zero the coefficients of lowest powers of ξ (i.e. $\xi^{s-2} = 0$) in equation (12), we obtain

$$\begin{bmatrix} (s^2 - b_3^2) & m^2(c_3^* s - b_1^2) & 0 \\ -(c_3^* s + b_1^2) & (c_1^* s^2 - b_2^2) & 0 \\ 0 & 0 & (s^2 - b_4^2) \end{bmatrix}\begin{bmatrix}\mathbf{A_0} \\ \mathbf{B_0} \\ \mathbf{C_0}\end{bmatrix} = 0 \qquad (15)$$

The equation (15) leads to non-trivial solution of the following two indicial equations

$$s^4 - A s^2 + C = 0 \ , \quad s^2 = b_4^2 \qquad (16)$$

where $A = \left((b_2^2 + \delta^2 b_3^2) - m^2\ (1-\delta^2)^2\right)/\,\delta^2$, $C = \left(b_2^2 b_3^2 - m^2 b_1^4\right)/\,\delta^2$,

The roots of equation (16) are of the type $s = \pm s_i\ (i = 1,\,2,\,3)$

$$s_1 = \sqrt{(A + \sqrt{A^2 - 4C})/2},$$
$$s_2 = \sqrt{(A - \sqrt{A^2 - 4C})/2} \ , \quad s_3 = b_4 \qquad (17)$$

We designate the roots $s_i\ (i = 1\ to\ 6)$ as

$$s_4 = -s_1 \ , \ s_5 = -s_2 \ , s_6 = -s_3 \ .$$

$$Y_0(s_1) = [1 \quad Q_B(s_1) \quad 0]'\,L_0,\ Y_0(s_2) = [1 \quad Q_B(s_2) \quad 0]'\,L_0,$$
$$Y_0(s_3) = [0 \quad 0 \quad 1]'\,L_0$$

$$Y_0(s_4) = [1 \quad Q_B(-s_1) \quad 0]'\,L_0,$$
$$Y_0(s_5) = [1 \quad Q_B(-s_2) \quad 0]'\,L_0,\ Y_0(s_6) = Y_0(-s_3)$$

where

$$Q_B(s_j) = -\frac{(s_j^2 - b_3^2)}{m^2\left((1-\delta^2)s_j - b_1^2\right)} \ ; \ Q_B(-s_j) = \frac{(s_j^2 - b_3^2)}{m^2\left((1-\delta^2)s_j + b_1^2\right)},$$

L_0 is a constant. It gives us an idea to write

$$\mathbf{A_0}(s_j) = \begin{cases} 1, & j = 1,2 \\ 0, & j = 3 \end{cases},\ \mathbf{B_0}(s_j) = \begin{cases} Q_B(p_j), & j = 1,2 \\ 0, & j = 3 \end{cases},$$

$$\mathbf{C_0}(s_j) = \begin{cases} 0, & j = 1,2 \\ 1, & j = 3 \end{cases} \ ; \qquad (18)$$

where $\mathbf{A_0}(s_j)$, $\mathbf{B_0}(s_j)$, $\mathbf{C_0}(s_j)$; $j = 4, 5$, 6 can be written from (18) by replacing s_j with $-s_j$,$(j = 1,\,2,\,3)$ and again equating to zero the coefficients of next lowest degree term ξ^{s-1}, which corresponds to $k = 1$, and using equations (18) we get:

$$\mathbf{G_1}(s_j+1)\mathbf{Y_1}+\mathbf{G_2}(s_j)\mathbf{Y_0}=0\ ,\ (j=1,2,3) \quad (19)$$

$$\mathbf{Y_1}\ =-\left(\mathbf{G_1}(s_j+1)\right)^{-1}\mathbf{G_2}(s_j)\mathbf{Y}_0=\mathbf{D_1^*}\mathbf{Y_0} \quad (20)$$

The matrix $\mathbf{D_1^*}$ is shown in Appendix (A 1.1). Now equating the coefficients of powers of $\xi^{s+k}=0,$ we have following recurrence relation:

$$\mathbf{G_1}(s_j+k+2)\mathbf{Y_{k+2}}+\mathbf{G_2}(s_j+k+1)\mathbf{Y_{k+1}}+\mathbf{G}\,\mathbf{Y_k}=0 \quad (21)$$

$$\mathbf{Y_{k+2}}=-\left(\mathbf{G_1}(s_j+k+2)\right)^{-1}\left[\mathbf{G_2}(s_j+k+1)\mathbf{Y_{k+1}}+\mathbf{G}\,\mathbf{Y_k}\right] \quad (22)$$

Now putting $k=0,1,2,3...$ successively, and simplifying we get

$$\mathbf{Y_{k+2}}==\mathbf{D^*_{k+2}}\mathbf{Y_0}$$

where

$$\mathbf{D^*_{k+2}}=-\left(\mathbf{G_1}(s_j+k+2)\right)^{-1}\left[\mathbf{G_2}(s_j+k+1)\mathbf{D^*_{k+1}}+\mathbf{G}\,\mathbf{D^*_k}\right]\ ,$$

5. Convergence Analysis

Analytical results show that the matrix $\mathbf{D^*_{k+2}}$ has similar form to that of $\mathbf{G_1}(s_j+k+2)$ for even values of k and it is alike $\mathbf{G_2}(s_j+k+1)$ for odd values of k. Thus we have

$$\mathbf{Y_{2k+2}}=\mathbf{D^*_{2k+2}}\mathbf{Y_0}\ ,\ \mathbf{Y_{2k+1}}=\mathbf{D^*_{2k+1}}\mathbf{Y_0}\ ,\ k=0,\ 1,\ 2,\ 3,...$$

where

$$\mathbf{D^*_{2k+2}}=-\left(\mathbf{G_1}(s_j+2k+2)\right)^{-1}\left[\mathbf{G_2}(s_j+2k+1)\,\mathbf{D^*_{2k+1}}+\mathbf{G}\,\mathbf{D^*_{2k}}\right],$$

$$\mathbf{D^*_{2k+1}}=-\left(\boldsymbol{G_1}(s_j+2k+1)\right)^{-1}\left[\mathbf{G_2}(s_j+2k)\mathbf{D^*_{2k}}+\mathbf{G}\,\boldsymbol{D^*_{2k-1}}\right], \quad (23)$$

According to Cullen [14] the matrix sequence $\{E_k\}$ in $C_{k\times k}$, we have $\lim\limits_{k\to\infty} E_k=E(\{E_k\}\to E)$ if each of the k^2 component sequence converges. Simplifications of equation (23) we have

$$\mathbf{D^*_{2k+2}}=\begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & 0 \\ 0 & 0 & H_{31} \end{bmatrix}\approx o\left(k^{-2}\right)E^* \quad ,$$

$$\mathbf{D^*_{2k+1}}=\begin{bmatrix} 0 & 0 & H'_{13} \\ 0 & 0 & H'_{23} \\ H'_{31} & H'_{32} & 0 \end{bmatrix}\approx o\left(k^{-1}\right)E^{**};$$

where $E^*==c_4^*\,diag\,(\Omega\overline{\beta}\,\hat{\tau}_1\ ,\ 0\ ,\ 4)$ and E^{**} is a zero matrix. The above values of H_{ij} and H'_{ij} are shown in Appendix (A 1.2) to (A 1.10). Using above facts, this noticed that both the matrices $\mathbf{D^*_{2k+2}}\to 0$, $\mathbf{D^*_{2k+1}}\to 0$, as $k\to\infty$. Thus the series having infinite radius of convergence and equation (11) is absolutely and uniformly convergent. Therefore the considered series can be differentiated term by term and also being analytic function.

6. Solution to obtain displacement and Stresses

The potential functions w, G, Θ and ψ obtained from equation (7) using equation (8) to (9) we get

$$\{w\,,G\,,\Theta\}=(\zeta)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\sum_{j=1}^{6}\sum_{k=0}^{\infty}C_{mjk}\begin{Bmatrix} a_{jk}(s_j)\,, \\ b_{jk}(s_j)\,, \\ d_{jk}(s_j) \end{Bmatrix}(\zeta\Omega)^{s_j+k}\ e^{-i(m\phi+\Omega\tau)} \quad (24)$$

$$\psi(\zeta,\ \phi,\ t)=(\zeta)^{-\frac{1}{2}}\begin{bmatrix} \sum\limits_{m=1}^{\infty}B_{m1}\ J_\eta\left(\dfrac{\zeta\Omega}{\delta}\right)+ \\ \sum\limits_{m=1}^{\infty}B_{m2}\ Y_\eta\left(\dfrac{\zeta\Omega}{\delta}\right) \end{bmatrix} e^{-i(m\phi+\Omega\tau)} \quad (25)$$

where $s_j\ (j=1\ ,\ 2\ ,\ 3,\ 4\ ,\ 5,\ 6)$ are the eigen-values and $\{\,a_{jk}(p_j),b_{jk}(p_j),d_{jk}(p_j)\,\}$ are eigenvectors corresponding to the eigen-values s_j and integer k. Here C_{mjk} are arbitrary constants to be evaluated by using the boundary conditions. The quantities $a_{jk}(s_j)$, $b_{jk}(s_j)$ and $d_{jk}(s_j)$ are defined as under:

$$a_{jk}(s_j)=\begin{cases} d_{11}^k(s_j)+d_{12}^k(s_j)Q_B(s_j), & j=1,2\ ;\ k=\text{even} \\ d_{13}^k(s_j) & ,\quad j=3\ ;\ k=\text{odd} \end{cases},$$

$$b_{jk}(s_j)=\begin{cases} d_{21}^k(s_j)+d_{22}^k(s_j)Q_B(s_j) & ,\ j=1,2\ ;\ k=\text{even} \\ d_{23}^k(s_j) & ,\ j=3\quad ;\ k=\text{odd} \end{cases},$$

$$d_{jk}(s_j)=\begin{cases} d_{31}^k(s_j)+d_{32}^k(s_j)Q_B(s_j) & ,\ j=1,2\ ;\ k=\text{odd} \\ d_{33}^k(s_j) & ,\ j=3\ ;\ k=\text{even} \end{cases},$$

where $d_{ij}^k(s_j)\,;(i,j=1,2,3)$ are the elements of determinant $\mathbf{D_k^*}$. The unknowns B_{m1}, B_{m2} and C_{mjk} , $j=1,2,3,4,5,6)$ can be evaluated by four boundary conditions of spherical curved plate. The non dimensional displacements, stresses and temperature change are obtained as below

7. Secular Equations: By applying the boundary conditions of above equations (4), we get a system of eight homogeneous linear algebraic equations which leads to non-trivial solution for thermally insulated and isothermal spherical curved plate.

$$U_\theta = (\zeta)^{-\frac{1}{2}} \left[im \sum_{m=1}^{\infty} \left[B_{m1} J_\eta \left(\frac{\zeta\Omega}{\delta} \right) + B_{m2} Y_\eta \left(\frac{\zeta\Omega}{\delta} \right) \right] \right] e^{-i(m\phi+\Omega\tau)} \quad (27)$$

$$U_\phi = (\zeta)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \left[im \left(\sum_{j=1}^{6} \sum_{k=1}^{\infty} C_{mjk} b_{jk}(s_j) (\zeta\Omega)^{s_j+k} \right) \right] e^{-i(m\phi+\Omega\tau)} \quad (28)$$

$$\tau_{\zeta\zeta} = \zeta^{-\frac{1}{2}} \left[\sum_{k=0}^{\infty} \sum_{j=1}^{6} \left\{ \begin{array}{l} \sum_{m=1}^{\infty} C_{mjk}\, m^2 (1-2\delta^2)\, b_{j(k+1)}(s_j) \\ + \sum_{m=0}^{\infty} C_{mjk} \left(s_j + k + \frac{5}{2} - 4\delta^2 \right) \times \\ a_{j(k+1)}(s_j) + c_2^* d_{jk}(s_j) \end{array} \right\} (\zeta\Omega)^{s_j+k} \right] e^{-i(m\phi+\Omega\tau)} \quad (29)$$

For $k > 0$, $m > 0$, the secular equations are

$$\left| \overline{\mathbf{R}}'_{ij} \right| = 0 , \quad (i, j = 1, 2, 3, 4, 5, 6) \quad (33)$$

$$\overline{\mathbf{R}}'_{77} \overline{\mathbf{R}}'_{88} - \overline{\mathbf{R}}'_{87} \overline{\mathbf{R}}'_{78} = 0 \quad (34)$$

where

$$\overline{\mathbf{R}}'_{11} = \begin{bmatrix} \left(s_1 + k + \frac{1}{2} + 2(1-2\delta^2) \right) a_{j(k+1)}(s_1) + \\ m^2 (1-2\delta^2) b_{j(k+1)}(s_1) + c_2^* d_k(s_1) \end{bmatrix} (a\Omega)^{s_1+k}$$

$$\overline{\mathbf{R}}'_{31} = \begin{cases} \left(s_1 + k + \frac{1}{2} \right) d_{j(k+1)}(s_1) (a\Omega)^{s_1+k} ; & \text{for insulated} \\ d_k(s_1) (a\Omega)^{s_1+k} ; & \text{for isothermal} \end{cases}$$

$$\overline{\mathbf{R}}'_{51} = \begin{bmatrix} a_{j(k+1)}(s_1) + \\ \left(s_1 + k - \frac{1}{2} \right) b_{j(k+1)}(s_1) \end{bmatrix} (a\Omega)^{p_1+k} \quad (35)$$

$$\overline{\mathbf{R}}'_{77} = \left[\left(\eta - \frac{3}{2} \right) J_\eta (a_1 \Omega/\delta) - (a\Omega/\delta) J_{\eta+1} (a\Omega/\delta) \right]$$

The elements $\det(\overline{\mathbf{R}}'_{ij}), (i=2, 4, 6;\ j=2, 3, 4, 5, 6)$ can be obtained by just replacing s_1 in $\det(\overline{\mathbf{R}}'_{ij}), (i=1, 3, 5)$ with p_j , $(j=2, 3, 4, 5, 6)$ while $\det(\overline{\mathbf{R}}'_{ij}), (i=2, 4, 6)$ are obtained by replacing a in $\det(\overline{R}'_{ij}) = 0$, $(i=1, 3, 5)$; $j=1, 2, 3, 4, 5, 6$ with b . The element of $\overline{\mathbf{R}}'_{78}$ can be obtained by replacing Bessel's function of first kind J_η with that of second kind Y_η in equation (48) and that $\overline{\mathbf{R}}'_{87}$ and $\overline{\mathbf{R}}'_{88}$ are obtained from $\overline{\mathbf{R}}'_{77}$ and $\overline{\mathbf{R}}'_{78}$ by replacing a with b , respectively.

$$\tau_{\zeta\phi} = -(\zeta)^{-\frac{1}{2}} \left[\sum_{k=0}^{\infty} \sum_{j=1}^{6} \left\{ \begin{array}{l} \sum_{m=0}^{\infty} C_{mjk}\, a_{j(k+1)}(s_j) + \\ \sum_{m=0}^{\infty} C_{mjk} \left(\frac{1}{2} - s_j - k \right) \\ b_{j(k+1)}(s_j) \end{array} \right\} (\zeta\Omega)^{s_j+k} \right] e^{-i(m\phi+\Omega\tau)} \quad (31)$$

$$\Theta,_\zeta = \zeta^{-\frac{1}{2}} \left[\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{6} \frac{C_{mjk}(s_j + k + (1/2))}{d_{j(k+1)}(s_j)\, (\xi\Omega)^{s_j+k}} \right] e^{-(m\phi+\Omega\tau)} \quad (32)$$

where $\bar{J}(\zeta\Omega/\delta) = \{ \eta J_\eta (\zeta\Omega/\delta) - (\zeta\Omega/\delta) J_{\eta+1}(\zeta\Omega/\delta) \}$,

$\bar{Y}(\zeta\Omega/\delta) = \{ \eta Y_\eta (\zeta\Omega/\delta) - (\zeta\Omega/\delta) Y_{\eta+1}(\zeta\Omega/\delta) \}$,

Toroidal vibrations: Solving equation (34) by using asymptotic expansions of ψ from Ding et al. [13] becomes:

$$\frac{\tan\left((\zeta/\delta)\Omega h \right)}{(\zeta/\delta)\Omega h} \cong \frac{4\eta^2 + 15}{8ab\Omega^2 - 4\eta^2 + 33} \quad (36)$$

where $h = a - b$, $R = \frac{a+b}{2}$, $\eta^2 = \frac{9}{4} + m^2$

When the thickness h tending to zero, the equation (36) leads to:

$$R^2\Omega^2 \cong \eta^2 - 9/4 \quad (37)$$

The equations (36) and (37) are in good agreement with Cohen et al. [5] and discussed detail there in.

Spheroidal vibrations: The secular equations (33) and (35) govern the spheroidal vibrations (S-modes).

8. Numerical results and discussion: In order to study the analytical results we compute numerical calculations for lowest frequency of S-modes of polymethyl methacrylate material for $k > 0$, $m > 0$ with respect to thickness to mean radius ratio by using iteration numerical technique with the help of MATLAB software tools. The presence of dissipation term in heat conduction equation (4), the secular equations provide us complex values of the non-dimensional frequency Ω and hence of ω. If we write $\omega = \omega_R + i\omega_I$, the lowest frequency and dissipation factor are given by $\hat{\Omega} = \mathrm{Re}\left(\frac{\Omega}{\delta} \right) = \left(\frac{R\omega_R}{c_2} \right)$ and $D = \mathrm{Im}\left(\frac{\Omega}{\delta} \right) = \left(\frac{R\omega_R}{c_2} \right)$ for fixed values of m and k . By taking sufficient number of Fröbenius parameter k to obtain the converged values of lowest frequency (Ω_R) and dissipation factor (D). The numerically computed lowest frequency, dissipation factor have been presented in Figs. 1 to 12 for viscothermoelastic (VTE) material of spherical curved plate. The physical data of polymethyl methacrylate material is given as Othman et al. [8]

$\varepsilon_T = 0.045$, $\omega^* = 1.11 \times 10^{11} s^{-1}$, $T_0 = 773 K$,

$\delta^2 = 0.333$, $t_0 = 0.2$, $t_1 = 0.02$,

$K = 0.19 W m^{-1} K^{-1}$, $C_e = 1400 J kg^{-1} K^{-1}$,

$\alpha_T = 77 \times 10^{-6} K^{-1}$, $\rho = 1190 kg m^{-3}$,

The variations of lowest frequency (Ω_R) and dissipation factor (D) versus thickness to mean radial ratio (t^*) i.e. $(t^* = h/R)$, $h = a - b$ and $R = \frac{a+b}{2}$ have been presented in Figs. 1 to Fig. 2 for first three modes of wave numbers $(m = 1, 2, 3)$ for GTE (generalized thermoelasticity) and CTE (coupled thermoelasticity). It is concluded from Figs. 1 to Fig. 2 that the vibrations initially low attains maximum value at $(t^* = 0.2)$ and with increase in the value of (t^*) the vibrations go on decreasing and become asymptotic.

Figs. 3 and Fig. 4 have shown for the vibrations of wave number (m) versus frequency modes $(\Omega = 10)$ for fixed number of thickness to mean radial ratio t^* . It has been concluded that with increase in modes of frequency (Ω) the vibrations go on increasing.

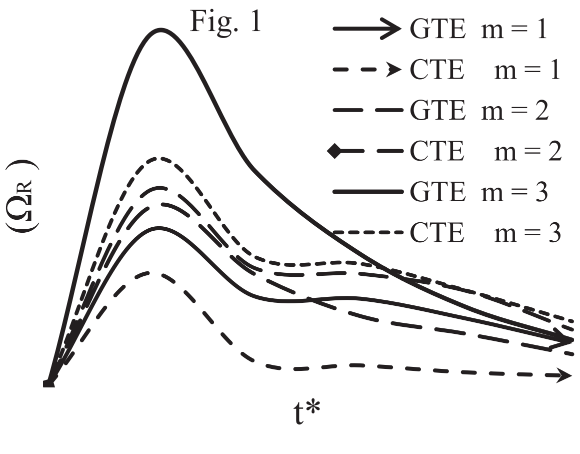

Fig. 1. Lowest frequency versus thickness to mean radial ratio t^* for $m = 1 , 2 , 3$ for spheroidal vibrations.

It is observed that the trends of variation of wave number (m) versus modes of frequency (Ω) decreases as thickness to mean radial ratio increases. It is inferred from the Fig.5 in which the lowest frequency versus wave number (m) of toroidal vibrations have been presented for thickness to mean radial ratio $(t^* = 0.11, 0.22, 0.37, 0.50, 0.66, 0.85)$ that there is gradual increase in the lowest frequency of vibrations up to $(m = 2)$, after that with the increasing value of (m), the behavior of vibrations become linear and stable.

The trends of the profiles in Figs. 5 are similar to that as reported in Ding et al. [13]. But the dissipation factor in the toroidal vibrations is very low (of the order 10^{-9}) in the instant case which is negligible. From the trends of variations of lowest frequency and dissipation factor of S – mode, it is noticed that the

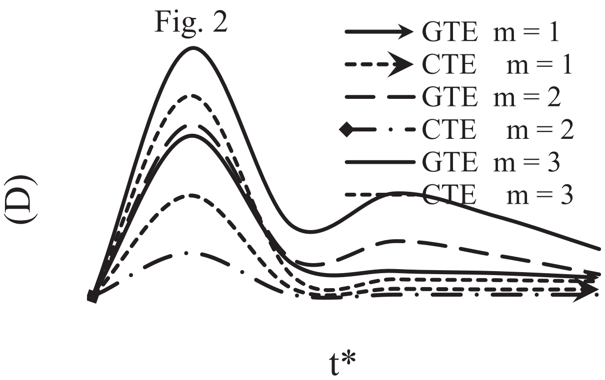

Fig. 2. Dissipation factor versus thickness to mean radial ratio t^* for $m = 1 , 2 , 3$ for spheroidal vibrations.

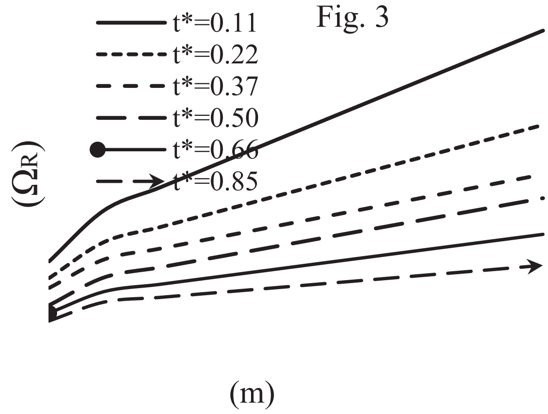

Fig. 3. Lowest frequency versus wave number (m) for different values of thickness to mean radial ratio t^* for spheroidal vibrations.

thermal variations, thermal relaxation time significantly affect the characteristics and behavior of spherical vibrations and their magnitudes in contrast to that of toroidal modes which are not affected by the temperature variations as expected.

Conclusion: With the help of Helmholtz decomposition theorem and the Matrix Fröbenius method is successfully used to solve exactly the resulting system of equations. Due to temperature variations, dissipation energy is caused because of the damping term present in the heat conduction equation. As thickness to mean radial ratio increases the variation of vibrations of lowest frequency and dissipation factor of vibrations go on deceasing. This also depicts the thermal relaxation effects. The deformation of spherical curved plates depends on nature of tractions applied as well as boundary conditions. Convergence analysis shows that the components of the considered series are continuous and analytic in nature. The results are consistent with existing literature. This study may find applications in tribology and other geophysical industries.

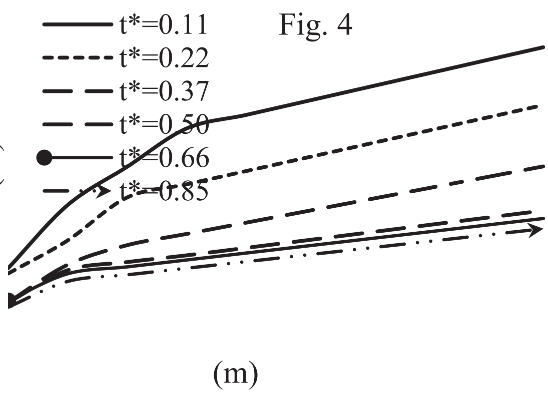

Fig. 4. Dissipation factor versus wave number (m) for different values of thickness to mean radial ratio t^* for spheroidal vibrations.

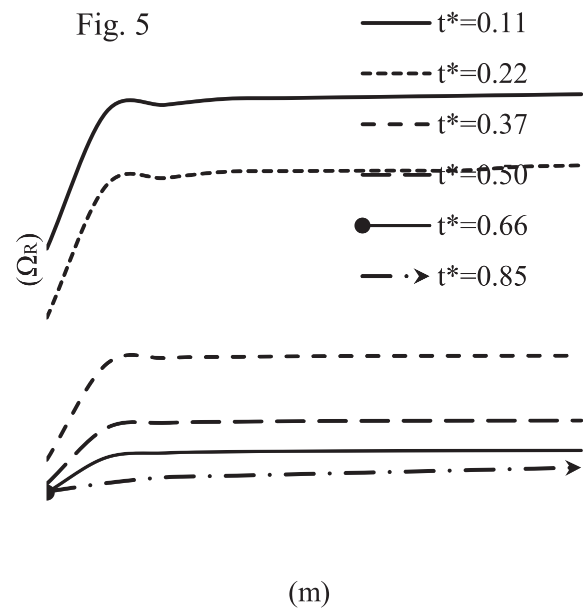

Fig. 5. Lowest frequency versus wave number (m) for different values of thickness to mean radial ratio t^* for toroidal vibrations.

Appendix:

$$\mathbf{D}_1^* = \begin{bmatrix} 0 & 0 & M_{23} \\ 0 & 0 & M_{23} \\ M_{31} & M_{31} & 0 \end{bmatrix} \qquad (A.1)$$

$$M_{31} = \frac{G'_{31}(s_j)}{G_{33}(s_j+1)} \quad , \quad M_{31} = \frac{G'_{32}(s_j)}{G_{33}(s_j+1)}$$

$$M_{13} = \frac{G_{22}(s_j+1)G'_{13}(s_j) - G'_{12}(s_j+1)H'_{23}(s_j)}{G_{11}(s_j+1)G_{22}(s_j+1) - G_{12}(s_j+1)G_{21}(s_j+1)}$$

$$M_{23} = \frac{G_{21}(s_j+1)G'_{13}(s_j) - G_{11}(s_j+1)G'(s_j)}{G_{11}(s_j+1)H_{22}(s_j+1) - G_{12}(s_j+1)G_{21}(s_j+1)}$$

$$H_{11} = \frac{G'_{31}(s_j+2k)G'_{13}(s_j+2k+1)}{G_{33}(s_j+2k+1)} + \Omega^2 \frac{G_{22}(s_j+2k)}{H^*(s_j+2k)} \quad ; \qquad (A.2)$$

$$H_{12} = \frac{G'_{32}(s_j+2k)G'_{13}(s_j+2k+1)}{G_{33}(s_j+2k+1)} + \Omega^2 \frac{G_{12}(s_j+2k)}{H^*(s_j+2k)} \quad ; \qquad (A.3)$$

$$H_{21} = \frac{G'_{31}(s_j+2k)G'_{23}(s_j+2k+1)}{G_{33}(s_j+2k+1)} + \Omega^2 \frac{G_{21}(s_j+2k)}{H^*(s_j+2k)} \quad ; \qquad (A.4)$$

$$H_{22} = \frac{G'_{32}(s_j+2k)G'_{23}(s_j+2k+1)}{G_{33}(s_j+2k+1)} + \Omega^2 \frac{G_{11}(s_j+2k)}{H^*(s_j+2k)} \quad ; \qquad (A.5)$$

$$H_{33} = \frac{G'_{31}(s_j + 2k + 1)K_1^*}{H^*(s_j + 2k + 1)} + \frac{G'_{32}(s_j + 2k + 1)K_2^*}{H^*(s_j + 2k + 1)} + \frac{\hat{\tau}_0^{*\prime}}{G_{33}(s_j + 2k)} \quad ; \quad (A.6)$$

$$K_1^* = \begin{pmatrix} G_{22}(s_j + 2k + 1)G'_{13}(s_j + 2k) \\ - G_{12}(s_j + 2k + 1)G'_{23}(s_j + 2k) \end{pmatrix} \quad ;$$

$$K_2^* = \begin{pmatrix} G_{21}(s_j + 2k + 1)G'_{13}(s_j + 2k) \\ + G_{11}(s_j + 2k + 1)G'_{23}(s_j + 2k) \end{pmatrix} \quad ;$$

$$H^*(s_j + 2k) = \begin{pmatrix} G_{11}(s_j + 2k)G_{22}(s_j + 2k) \\ - G_{21}(s_j + 2k)G_{12}(s_j + 2k) \end{pmatrix};$$

$$H'_{13} = \frac{G'_{13}(s_j + 2k)}{G_{33}(s_j + 2k)} + \Omega^2 \frac{K_1^*}{H^*(s_j + 2k + 1)} \quad ; \quad (A.7)$$

$$H'_{23} = \frac{G'_{23}(s_j + 2k)}{G_{33}(s_j + 2k)} + \Omega^2 \frac{K_2^*}{H^*(s_j + 2k + 1)} \quad ; \quad (A.8)$$

$$H'_{31} = \frac{K_3^*}{H^*(s_j + 2k)} - \frac{G'_{31}(s_j + 2k + 1)\hat{\tau}_0^{*\prime}}{G_{33}(s_j + 2k + 1)} \quad ; \quad (A.9)$$

$$H'_{32} = \frac{K_4^*}{H^*(s_j + 2k)} + \frac{G'_{32}(s_j + 2k + 1)\hat{\tau}_0^{*\prime}}{G_{33}(s_j + 2k + 1)} \quad ; \quad (A.10)$$

$$K_3^* = \begin{pmatrix} - G'_{31}(s_j + 2k)G_{22}(s_j + 2k) + \\ G'_{31}(s_j + 2k)G_{12}(s_j + 2k) \end{pmatrix}$$

$$K_4^* = \begin{pmatrix} - G'_{31}(s_j + 2k)\,G_{21}(s_j + 2k) \\ + G'_{31}(s_j + 2k)\;G_{11}(s_j + 2k) \end{pmatrix}$$

References:

1. R. S. Dhaliwal and A. Singh, "Dynamic Coupled Thermo elasticity", New Delhi: Hindustan Publication Corporation, (1980).
2. H. Lamb, "On the vibrations of a elastic sphere", Proc. London Math. Soc., 13 (1882), 189-212.
3. Y. Sato and T. Usam i, "Basic study on the oscillation of homogeneous elastic sphere part-II, distribution of displacement", Geophysics Magazine, 31 (1962), 25-47.
4. W. Q. Chen, J. B. Cai, G. R. Ye and H. J. Ding, "On Eigen frequencies of an Anisotropic Sphere", J. Appl. Mech. ASME. 67 No. 2 (2000), 422 – 424.
5. H. Cohen, A. H. Shah and C. V. Ramakrishan, "Free Vibrations of a Spherically Isotropic Hollow Sphere", Acustica, 26 (1972), 329 – 333.
6. H. W. Lord and Y. Shulman, "A generalization of dynamical theory of thermoelasticity", J. Mech. Physics Solids, 15, No. 1 (1967), 299 – 309.
7. E. Green and K. A. Lindsay, "Thermo-elasticity", J. Elasticity, 2, No. 1 (1972), 1 – 7.
8. M. I. A.Othman, M. A. Ezzat, S. A. Zaki and A.S. El-Karamany, "Generalized thermo-viscoelasctic plane waves with two relaxation times", Int. J. Engng Sci, 40, No. 12 (2002), 1329 – 1347.
9. Yu. Jiangong, and M. A. Qiujuan, "Wave characteristics in magneto-electro-elastic functionally graded spherical curved plates", Mech. Advan. Mater. Struct, 17 (2010),
10. S.Towfighi, T. Kundu and M. "Ehsani, Elastic wave propagation in circumferential direction in anisotropic cylindrical curved plates", J. Appl. Mech, ASME., 69 (2002) 283–291.
11. J. N. Sharma and N. Sharma, "Three dimensional Free Vibration Analysis of a Homogeneous transradially Isotropic thermo-elastic Sphere", J. Appl. Mech, ASME, 77, No. 2 (2010) 021004-1-9.
12. S. Towfighi and T. Kundu, "Elastic wave propagation in anisotropic spherical curved plates", Int. J. Solids Struct., 40, No. 20 (2003), 5495–5510.
13. H. Ding, W. Chen and L. "Zhang, Elasticity of Transversally Isotropic Materials", The Netherlands, Springer (2006).
14. C. G. Cullen, "Matrices and Linear transformations", Palo Alto London: Addison-Wesley Publishing Company Reading Massachusetts, (1966).

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